Prepotentials of $N = 2$ $SU(2)$ Yang-Mills Theories Coupled with Massive Matter Multiplets

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Abstract

We discuss $N = 2$ $SU(2)$ Yang-Mills gauge theories coupled with N_f $(= 2, 3)$ massive hypermultiplets in the weak coupling limit. We determine the exact massive prepotentials and the monodromy matrices around the weak coupling limit. We also study that the double scaling limit of these massive theories and find that the massive $N_f - 1$ theory can be obtained from the massive N_f theory. New formulae for the massive prepotentials and the monodromy matrices are proposed. In these formulae, N_f dependences are clarified.

I. INTRODUCTION

Non-perturbative properties of four dimensional $N=2$ supersymmetric $SU(2)$ Yang-Mills gauge theory was discussed by Seiberg and Witten[1,2]. One of the important discoveries in their investigations was the fact that the quantum moduli space of the $N=2$ $SU(2)$ Yang-Mills theory coupled with or without N_f hypermultiplets could be identified with the moduli spaces of certain elliptic curves which controlled the low energy properties. They could determine the exact expressions for the monopole and dyon spectrum and the metric on the quantum moduli space. Their approach was extended to, for example, the other gauge theory with or without matters[3–8]. In Ref.9, the quantum moduli space of $N=2$ $SU(2)$ Yang-Mills theories coupled with mass-less hypermultiplets was studied, but our knowledges for the massive theories are poor in contrast with the case of the mass-less theories. Of course Seiberg and Witten qualitatively discussed these massive theories in Ref.2, but we can not say that we have enough quantitative understandings for these massive theories because the quantitative analyses on them does not ever been sufficiently carried out.

For this reason, we discussed the simplest massive theory, i.e, $N_f = 1$ theory, in the weak coupling limit as an instructive example[10]. In Ref.10, we did not discuss the other asymptotic free theories, i.e., $N_f = 2$ and 3 because there were several technical obstacles in the computations of the periods. However, for the massive $N_f = 1$, we could find the exact prepotential and the monodromy matrix by using Picard-Fuchs equation[10]. We observed that the Picard-Fuchs equation was a third order differential equation. Its solutions could not be expressed by a hypergeometric function in contrast with the massless theory but they gave interesting informations for the massive $N_f = 1$ theory. For example, the $N_f = 0$ theory can be regarded as a low energy version of the massive $N_f = 1$ theory[2] and it must be obtained from the massive $N_f = 1$ theory in the double

scaling limit, but we could explicitly show how the massive $N_f = 1$ theory flowed to the $N_f = 0$ theory by using those solutions. On the other hand, since the massive $N_f = 1$ theory can be regarded as a low energy theory of the massive $N_f = 2$ theory, we are sure that the results of Ref.10 can be obtained from the massive $N_f = 2$ theory in the double scaling limit. In general, the massive $N_f - 1$ theory can be considered as a low energy theory of the massive N_f theory. Therefore in order to determine the general structure of these massive gauge theories, we must extend the results of Ref.10. For these reasons, we discuss the massive $N_f = 2$ and 3 theories in the weak coupling limit in this paper. In the text, we mainly discuss the massive $N_f = 2$ theory because all mathematical expressions in the massive $N_f = 3$ theory are lengthy. Therefore, we summarize the results for the massive $N_f = 3$ theory in appendix C.

The paper consists of the following sections. In section 2, we derive the Picard-Fuchs equation of the massive $N_f = 2$ theory and its solutions in the weak coupling limit. The order of the differential equation is three as well as that of the massive $N_f = 1$ equation. We discuss the monodromy around the weak coupling limit in the end of this section. The monodromy matrix can be arranged to 3×3 matrix due to the order of the differential equation. In addition to this, we will see that the monodromy matrix should be quantized by the winding numbers for the residues. We derive the prepotential and instanton contributions for it in section 3. The double scaling limit of this theory is discussed in section 4. We can find that the instanton expansion coefficients of the prepotential are completely coincide with that of Ref.10 in the double scaling limit. Final section 5 is summary. We summarize our results as some useful formulae for $N = 2$ $SU(2)$ Yang-Mills gauge theories weakly coupled with N_f ($N_f = 0, \cdots, 3$) matter multiplets. In particular, general formulae of the two periods of the meromorphic 1-form, the monodromy matrices and the exact prepotentials are proposed. We emphasis that N_f dependences are clarified by these results.

II. $N_F = 2$ PICARD-FUCHS EQUATION

Quantum moduli space of the $N = 2$ $SU(2)$ Yang-Mills theory coupled with two massive hypermultiplets can be described by the following hyperelliptic curve

$$y^2 = \left(x^2 - u + \frac{\Lambda_2^2}{8}\right)^2 - \Lambda_2^2(x + m_1)(x + m_2) \tag{2.1}$$

and the meromorphic 1-form[8]

$$\lambda_2 = \frac{\sqrt{2}xdx}{4\pi iy}\left[\frac{(x^2 - u + \Lambda_2^2/8)(2x + m_1 + m_2)}{2(x + m_1)(x + m_2)} - 2x\right], \tag{2.2}$$

where u is the gauge invariant parameter, m_1 and m_2 are masses of the hypermultiplets and Λ_2 is a dynamically generated mass scale of the theory. We can also describe the same $N_f = 2$ theory by an elliptic curve[2].

This curve has four branching points. In the weak coupling limit ($u \longrightarrow \infty$), they will behave as

$$\begin{aligned}
x_1 &= -\frac{\Lambda_2}{2} - \sqrt{u} + \frac{1}{\sqrt{u}}\left[-\frac{\Lambda_2^2}{16} + \frac{\Lambda_2}{4}(m_1 + m_2)\right] + \frac{\Lambda_2}{16u}(-m_1 + m_2)^2 + \cdots, \\
x_2 &= -\frac{\Lambda_2}{2} + \sqrt{u} + \frac{1}{\sqrt{u}}\left[\frac{\Lambda_2^2}{16} - \frac{\Lambda_2}{4}(m_1 + m_2)\right] + \frac{\Lambda_2}{16u}(-m_1 + m_2)^2 + \cdots, \\
x_3 &= \frac{\Lambda_2}{2} - \sqrt{u} + \frac{1}{\sqrt{u}}\left[-\frac{\Lambda_2^2}{16} - \frac{\Lambda_2}{4}(m_1 + m_2)\right] - \frac{\Lambda_2}{16u}(-m_1 + m_2)^2 + \cdots, \\
x_4 &= \frac{\Lambda_2}{2} + \sqrt{u} + \frac{1}{\sqrt{u}}\left[\frac{\Lambda_2^2}{16} + \frac{\Lambda_2}{4}(m_1 + m_2)\right] - \frac{\Lambda_2}{16u}(-m_1 + m_2)^2 + \cdots.
\end{aligned} \tag{2.3}$$

We can take the cuts to run counter clockwise from x_3 to x_1 as α-cycle and from x_3 to x_4 as β-cycle. The intersection number is $\alpha \cap \beta = 1$. We can regard the curve as a genus one Riemann surface.

The period integrals of λ_2 are defined by

$$a_2(u) = \oint_\alpha \lambda_2, \tag{2.4}$$

$$a_D^2(u) = \oint_\beta \lambda_2. \tag{2.5}$$

$a_2(u)$ is identified with the scalar component of the $N=1$ chiral multiplet and $a_D^2(u)$ is its dual[1,2]. In the mass-less theory, the above α and β constitute canonical homology bases of the torus, but in the massive theory they should be replaced with loops so as to enclose the "extra" poles corresponding to $x=-m_1,-m_2$. We will often use $\Pi_2=\oint_\gamma \lambda_2$, where γ is a suitable 1-cycle on the curve, as a representative of the periods.

It is more convenient to study the Picard-Fuchs equation than the period integrals themselves in order to see the behaviour near $u=\infty$. The massive $N_f=2$ Picard-Fuchs equation is a third order differential equation which is given by

$$\frac{d^3\Pi_2}{du^3}+\left(\frac{\Delta_2'}{\Delta_2}-\frac{8D_2}{B_2}\right)\frac{d^2\Pi_2}{du^2}-\frac{16}{\Delta_2}\left(C_2-\frac{A_2D_2}{B_2}\right)\frac{d\Pi_2}{du}=0, \tag{2.6}$$

where

$$\begin{aligned}
A_2 &= -\Lambda_2^6-8\Lambda_2^4\left[3(m_1^2+m_2^2)-u-m_1m_2\right]+256u^2(m_1^2+m_2^2-2u)\\
&\quad +32\Lambda_2^2\left[2u^2+4m_1^2m_2^2+14m_1m_2u-(m_1^2+m_2^2)(3u+6m_1m_2)\right],\\
B_2 &= 8\Lambda_2^4-64\Lambda_2^2m_1m_2+256\left[3u(m_1^2+m_2^2)-2u^2-4m_1^2m_2^2\right],\\
C_2 &= \Lambda_2^4-4\Lambda_2^2\left[3(m_1^2+m_2^2)-4u-12m_1m_2\right]\\
&\quad +32\left[3u(m_1^2+m_2^2)-6u^2-2m_1^2m_2^2\right],\\
D_2 &= 32\left[3(m_1^2+m_2^2)-4u\right],\\
\Delta_2 &= \Lambda_2^8-48\Lambda_2^6m_1m_2-16\Lambda_2^4\left[27(m_1^4+m_2^4)-36u(m_1^2+m_2^2)\right.\\
&\quad \left.+8u^2+6m_1^2m_2^2\right]+512m_1m_2\Lambda_2^2\left[9u(m_1^2+m_2^2)-10u^2-8m_1^2m_2^2\right]\\
&\quad +4096u^2(m_1^2-u)(m_2^2-u),
\end{aligned} \tag{2.7}$$

and $\Delta_2'=d\Delta_2/du$. Δ_2 coincides with the discriminant of the curve. (2.6) can be obtained from the massive $N_f=3$ Picard-Fuchs equation in the double scaling limit. See appendix C. Note that (2.6) does not show any symmetry over the u-plane. From (2.6) and (2.7), we can easily find that this differential equation has regular singular points corresponding

to $\Delta_2 = 0$ and $B_2 = 0$. We are also interested in the property near the discriminant loci which correspond to extra mass-less states, but the calculations near such singularities are not so easy, although they should be discussed elsewhere.

When both of the hypermultiplets have zero mass, (2.6) exactly reduces to

$$(\Lambda_2^4 - 64u^2)\frac{d^2\Pi_2}{du^2} - 16\Pi_2 = 0. \tag{2.8}$$

We set the integration constant to zero because (2.8) can be directly obtained using mass-less meromorphic 1-form. The global $\boldsymbol{Z}_2$ symmetry over the u-plane is now recovered. This symmetry can appear when and only when both of the matters are mass-less. Therefore we can find that the masses play a role to break the global symmetry. (2.8) was studied in Ref.9.

The reader may notice that the order of (2.6) is three whereas that of (2.8) is two. The mathematical background of this fact was discussed in Ref.10 in the case of the massive $N_f = 1$ theory, so we briefly state here only the essence. First, recall that the massive meromorphic 1-form has extra simple poles corresponding to $x = -m_1$ and $-m_2$. Since an operation of differentiating over u and integrating over x reduces the order of poles by one, the reduction will require one step more than the mass-less case when λ_2 is massive. Therefore the order of (2.6) and (2.8) differs by one. For more complete treatment, see Refs.10 and 11.

Next, let us try to calculate the solutions to (2.6). In order to accomplish it in the weak coupling limit, we introduce $z = 1/u$. We find that the solutions to the indicial equation for (2.6) are $0, -1/2, -1/2$ (double roots). The solution $\rho_0(z)$ corresponding to the index 0 is some constant ϵ_2 which may depend on Λ_2, m_1 and m_2,

$$\rho_0(z) = \epsilon_2. \tag{2.9}$$

At first sight, this constant solution is trivial but it is important and has non-trivial

meanings in the massive theory, as was explicitly shown in Ref.10. In fact, this corresponds to the residue contributions of λ_2 and we can rewrite (2.9) as

$$\rho_0(z) = \text{linear combination of } \nu_1 \text{ and } \nu_2, \tag{2.10}$$

where ν_1 and ν_2 are residues of λ_2. These constants will be determined in the comparison of the lower order expansion of the period integrals with fundamental solutions to the Picard-Fuchs equation. On the other hand, there are two independent solutions corresponding to the index $-1/2$. One of them is

$$\rho_1(z) = z^{-1/2} \sum_{i=0}^{\infty} a_{2,i} z^i, \tag{2.11}$$

where the first several expansion coefficients $a_{2,i}$ are given in appendix A. We find that $a_{2,n}$ can be represented by a polynomial of $\Lambda_2^{2i} m_1^j m_2^k$ with $2n = 2i + j + k$, where i, j, and k are non-negative integers. ρ_1 coincides with a hypergeometric function in the mass-less limit[9]. The other solution behaves logarithmic. It is

$$\rho_2(z) = \rho_1(z) \ln z + z^{-1/2} \sum_{i=1}^{\infty} b_{2,i} z^i \tag{2.12}$$

where the first several coefficients $b_{2,i}$ are given in appendix B. $b_{2,n}$ can also be represented by a polynomial of $\Lambda_2^{2i} m_1^j m_2^k$ with $2n = 2i + j + k$ as well. Note that these polynomials are actually homogeneous and have obvious $\boldsymbol{Z}_2$ symmetry, i.e, invariance under $m_1 \leftrightarrow m_2$.

We can express the periods (2.4) and (2.5) as a linear combination of ρ_0, ρ_1 and ρ_2 by comparison with the lower order expansion of the period integrals. The results will be

$$a_2(u) = \frac{\rho_1(z)}{\sqrt{2}} + n_1 \nu_1 + n_2 \nu_2, \tag{2.13}$$

$$a_D^2(u) = A \rho_2(z) + B \rho_1(z) + n_1' \nu_1 + n_2' \nu_2, \tag{2.14}$$

where

$$
\begin{aligned}
A &= -\frac{i\sqrt{2}}{2\pi}, \\
B &= \frac{i\sqrt{2}}{2\pi}(-2 + 4\ln 2 + \pi i - 2\ln\Lambda_2), \\
\nu_1 &= -\frac{\sqrt{2}}{4}m_1, \\
\nu_2 &= -\frac{\sqrt{2}}{4}m_2.
\end{aligned} \tag{2.15}
$$

We have identified ν_1 and ν_2 in the comparison. Both of the periods are now "quantized" by the winding numbers n_i and n'_i. This fact is characteristic to massive theories.

To end this section, let us comment on the monodromy. From (2.13) and (2.14), we can easily find that the monodromy matrix $M_{2,\infty}$ around $u = \infty$ acts to the column vector $v_2 = {}^t(a_D^2, a_2, \epsilon_2)$ as $v_2 \longrightarrow M_{2,\infty} \cdot v_2$, i.e,

$$
\begin{pmatrix} a_D^2 \\ a_2 \\ \epsilon_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 2 & \mu_2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_D^2 \\ a_2 \\ \epsilon_2 \end{pmatrix}, \tag{2.16}
$$

where $\epsilon_2 = n_1\nu_1 + n_2\nu_2$ and $\mu_2 = 2(n'_1 - n_1)/n_1 = 2(n'_2 - n_2)/n_2$. Note that in the mass-less limit we can easily recover the monodromy of the mass-less $N_f = 2$ theory[9].

Finally, since the residue contributions can appear also in the periods near the regular singular points, the monodromy matrices near them will be quantized as well. Of course mass-less theories do not have such residue contributions, so we can conclude that the monodromy matrix can be quantized, in general, only in massive theories.

III. PREPOTENTIAL

We can obtain the prepotential $\mathcal{F}_2$ from the relation

$$
a_D^2 = \frac{d\mathcal{F}_2}{da_2}. \tag{3.1}
$$

For that purpose, we should express a_D^2 as a series of a_2. However, since a_2 has constant terms, it is convenient to use $\tilde{a}_2 = a_2 - n_1\nu_1 - n_2\nu_2$ as a new variable. Then a_D^2 will be expanded as

$$\begin{aligned} a_D^2 = n_1'\nu_1 + n_2'\nu_2 + \sqrt{2}\tilde{a}_2(B - A\ln 2 - 2A\ln\tilde{a}_2) + \frac{A}{\sqrt{2}}\Bigg\{ \frac{1}{2\tilde{a}_2}(m_1^2+m_2^2) \\ -\frac{1}{6144\tilde{a}_2^3}\left[3\Lambda_2^4 + 384\Lambda_2^2 m_1 m_2 - 256(m_1^4+m_2^4)\right] \\ + \frac{1}{30720\tilde{a}_2^5}\left[45\Lambda_2^4(m_1^2+m_2^2) + 256(m_1^6+m_2^6)\right] + \cdots \Bigg\}. \end{aligned} \quad (3.2)$$

Therefore the prepotential will be

$$\begin{aligned} \mathcal{F}_2 = i\frac{\tilde{a}_2^2}{\pi}\Bigg[\frac{1}{2}\ln\left(\frac{\tilde{a}_2}{\Lambda_2}\right)^2 + \left(-1 + \frac{i\pi}{2} + \frac{5}{2}\ln 2\right) - \frac{\sqrt{2}\pi}{4i\tilde{a}_2}(n_1'm_1 + n_2'm_2) \\ - \frac{\ln\tilde{a}_2}{4\tilde{a}_2^2}(m_1^2+m_2^2) + \sum_{i=2}^{\infty}\mathcal{F}_i^2\tilde{a}_2^{-2i}\Bigg], \end{aligned} \quad (3.3)$$

where the first few coefficients of the prepotential are given by

$$\begin{aligned} \mathcal{F}_2^2 &= -\frac{\Lambda_2^4}{8192} + \frac{1}{96}(m_1^4+m_2^4) - \frac{\Lambda_2^2}{64}m_1m_2, \\ \mathcal{F}_3^2 &= \frac{3\Lambda_2^4}{16384}(m_1^2+m_2^2) + \frac{1}{960}(m_1^6+m_2^6), \\ \mathcal{F}_4^2 &= -\frac{5\Lambda_2^8}{268435456} + \frac{1}{5376}(m_1^8+m_2^8) - \frac{5\Lambda_2^6}{393216}m_1m_2 - \frac{5\Lambda_2^4}{32768}m_1^2m_2^2. \end{aligned} \quad (3.4)$$

We find that these expansion coefficients have the same structure as $a_{2,n}$ or $b_{2,n}$. In the mass-less limit, we can easily recover the result of Ref.9.

The reader may notice that this massive prepotential contains a curious term proportional to $(m_1^2+m_2^2)\ln\tilde{a}_2$, but we can observe that

$$\begin{aligned} \mathcal{F}_s^2 &= \sum_{i=1}^{2}\left(\tilde{a}_2 - \frac{m_i}{\sqrt{2}}\right)^2\ln\left(\tilde{a}_2 - \frac{m_i}{\sqrt{2}}\right) + \sum_{i=1}^{2}\left(\tilde{a}_2 + \frac{m_i}{\sqrt{2}}\right)^2\ln\left(\tilde{a}_2 + \frac{m_i}{\sqrt{2}}\right) \\ &= (m_1^2+m_2^2)\left(\ln\tilde{a}_2 + \frac{3}{2}\right) + 4\tilde{a}_2^2\ln\tilde{a}_2 - \frac{m_1^4+m_2^4}{24\tilde{a}_2^2} - \frac{m_1^6+m_2^6}{240\tilde{a}_2^4} + \cdots. \end{aligned} \quad (3.5)$$

Therefore we can rewrite (3.3) as

$$\mathcal{F}_2 = i\frac{\tilde{a}_2^2}{\pi}\left[\frac{1}{2}\ln\left(\frac{\tilde{a}_2}{\Lambda_2}\right)^2 + \left(-1+\frac{i\pi}{2}+\frac{5}{2}\ln 2\right) - \frac{\sqrt{2}\pi}{4i\tilde{a}_2}(n_1'm_1+n_2'm_2)\right.$$
$$\left. + \ln\tilde{a}_2 + \frac{3}{8\tilde{a}_2^2}(m_1^2+m_2^2) - \frac{1}{4\tilde{a}_2^2}\mathcal{F}_s^2 + \sum_{i=2}^{\infty}\widetilde{\mathcal{F}}_i^2\tilde{a}_2^{-2i}\right], \tag{3.6}$$

where

$$\begin{aligned}
\widetilde{\mathcal{F}}_2^2 &= -\frac{\Lambda_2^4}{8192} - \frac{\Lambda_2^2}{64}m_1m_2,\\
\widetilde{\mathcal{F}}_3^2 &= \frac{3\Lambda_2^4}{16384}(m_1^2+m_2^2),\\
\widetilde{\mathcal{F}}_4^2 &= -\frac{5\Lambda_2^8}{268435456} - \frac{5\Lambda_2^6}{393216}m_1m_2 - \frac{5\Lambda_2^4}{32768}m_1^2m_2^2.
\end{aligned} \tag{3.7}$$

IV. DOUBLE SCALING LIMIT

In this section, we discuss the double scaling limit ($m_2 \longrightarrow \infty, \Lambda_2 \longrightarrow 0, m_2\Lambda_2^2 = \Lambda_1^3$ fixed) of the massive $N_f = 2$ theory. We may scale m_1 instead of m_2, holding $m_1\Lambda_2^2$ fixed. Since the low energy theory of the massive $N_f = 2$ can be regarded as the massive $N_f = 1$ theory[2], we can check the consistency of our calculation by the double scaling limit. Discussions on the double scaling limit for the massive $N_f = 1$ theory, i.e, reduction from the massive $N_f = 1$ to the $N_f = 0$ theory can be found in Ref.10.

First, let us discuss the Picard-Fuchs equation (2.6). In the double scaling limit, coefficients (2.7) will be

$$\begin{aligned}
A_2 &\longrightarrow A_{2\mathrm{dsl}} = 64m_2^2\cdot(4u^2-3m_1\Lambda_1^3),\\
B_2 &\longrightarrow B_{2\mathrm{dsl}} = 64m_2^2\cdot(12u-16m_1^2),\\
C_2 &\longrightarrow C_{2\mathrm{dsl}} = 32m_2^2\cdot(3u-2m_1^2),\\
D_2 &\longrightarrow D_{2\mathrm{dsl}} = 32m_2^2\cdot 3,\\
\Delta_2 &\longrightarrow \Delta_{2\mathrm{dsl}} = -16m_2^2\cdot\left[27\Lambda_1^6 - 32m_1\Lambda_1^3(9u-8m_1^2) - 256(m_1^2u^2-u^3)\right].
\end{aligned} \tag{4.1}$$

We can rewrite these coefficients to more convenient forms

$$\begin{aligned}
&A_{2\mathrm{dsl}} = -64m_2^2 A_1,\ B_{2\mathrm{dsl}} = -64m_2^2 B_1,\ C_{2\mathrm{dsl}} = -32m_2^2 C_1,\\
&D_{2\mathrm{dsl}} = -32m_2^2 D_1,\ \Delta_{2\mathrm{dsl}} = -16m_2^2 \Delta_1,
\end{aligned} \tag{4.2}$$

where A_1, etc are the coefficients of the massive $N_f = 1$ Picard-Fuchs equation. See appendix D. It is interesting to note that $A_{2\mathrm{dsl}}$, etc are m_2^2 multiples of A_1, etc , respectively. From (4.2), we find that (2.6) in the double scaling limit reduces to

$$\frac{d^3\Pi_2}{du^3} + \left(\frac{\Delta_1'}{\Delta_1} - 4\frac{D_1}{B_1}\right)\frac{d^2\Pi_2}{du^2} - \frac{32}{\Delta_1}\left(C_1 - \frac{A_1 D_1}{B_1}\right)\frac{d\Pi_2}{du} = 0. \tag{4.3}$$

This looks the Picard-Fuchs equation for the massive $N_f = 1$ theory[10]. However, since it is unclear whether Π_2 always reduces to Π_1, the period integral of the massive $N_f = 1$ theory, there is no assurance that the solution Π_2 to (4.3) equals to Π_1 exactly.

In fact, λ_2 in the double scaling limit will behave as

$$\begin{aligned}
\lambda_2 &\longrightarrow \frac{\sqrt{2}x dx}{4\pi i\tilde{y}}\left[\frac{x^2-u}{2}\left(\frac{1}{x+m_1} + \frac{1}{x+m_2}\right) - 2x\right]\\
&= \frac{\sqrt{2}x dx}{4\pi i\tilde{y}}\left[\frac{x^2-u}{2(x+m_1)} - 2x + \frac{x^2-u}{2m_2}\left(1 - \frac{x}{m_2} + \cdots\right)\right]\\
&= \lambda_1 + \frac{\sqrt{2}x dx}{4\pi i\tilde{y}}\cdot\frac{x^2-u}{2m_2}\left(1 - \frac{x}{m_2} + \cdots\right),
\end{aligned} \tag{4.4}$$

where $\tilde{y}^2 = (x^2-u)^2 - \Lambda_1^3(x+m_1)$ is the curve for the massive $N_f = 1$ theory and λ_1 is its meromorphic 1-form. We find that λ_2 in the double scaling limit consists of λ_1 and an extra 1-form. Of course, this extra 1-form under $m_2 \longrightarrow \infty$ vanishes, but we can expect that divergences related to large m_2 should appear in our solutions because we have calculated all quantities under the premise that Λ_2, m_1 and m_2 are finite. However we can drop the divergences because the heavy quark can be integrated out in taking the double scaling limit[2]. Then, Π_2 in (4.3) will be equal to Π_1 up to irrelevant divergences.

For example, for the residue contributions, we can eliminate "ν_2" which depends on m_2 while keeping ν_1. We can show that a_2 in the double scaling limit coincides with a_1, the corresponding period in the massive $N_f = 1$ theory. This fact means that a_2 does not

receive any effects originated from the extra 1-form. On the other hand, the effects are non-trivial for a_D^2. In order to extract them, it is better to rearrange it by a_2. Namely, we will be able to divide a_D^2 into the convergent part and the divergent one. For that purpose, it is convenient to use (3.2). Then we can actually see that the expansion coefficients in the double scaling limit are a sum of the finite part and the divergent one. Since this divergence depends only on m_2, we can eliminate it. In this case, a_D^2 coincides with the corresponding period a_D^1 of the massive $N_f = 1$ theory. In fact, we can easily find that we can obtain a_D^1 if we drop the m_2 dependences of (3.2) and rewrite it as a series of u, but the constants A and B which correspond to the initial conditions for the Picard-Fuchs equation should be replaced with those of the massive $N_f = 1$ theory[10].

Dropping these divergences and integrating a_D^2 over "a_1" to obtain the prepotential in the double scaling limit, we find that the instanton expansion coefficients are

$$\begin{aligned}
\widetilde{\mathcal{F}}_2^2 &\longrightarrow -\frac{1}{64}\Lambda_1^3 m_1, \\
\widetilde{\mathcal{F}}_3^2 &\longrightarrow \frac{3\Lambda_1^6}{16384}, \\
\widetilde{\mathcal{F}}_4^2 &\longrightarrow -\frac{5\Lambda_1^6 m_1^2}{32768}.
\end{aligned} \tag{4.5}$$

These are nothing other than the instanton expansion coefficients of the massive $N_f = 1$ theory[10]! The other asymptotic leading terms coincide with those of the massive $N_f = 1$ after the replacement of A and B, we do not write down them here. These can be easily obtained from (5.4) with $N_f = 1$ (see below). In this way, we can obtain the prepotential of the massive $N_f = 1$ theory.

V. SUMMARY

We have studied the moduli space of $N = 2$ $SU(2)$ Yang-Mills gauge theory coupled with $N_f = 2, 3$ massive matter multiplets and clarified the relation between the massive

theories by double scaling limit. Though we have mainly stated on the massive $N_f = 2$ theory in the previous sections, by using the results in appendix C, Refs.5 and 10, we can write the periods of the massive N_f ($N_f = 0, \cdots, 3$) theory as

$$\begin{aligned} a_{N_f}(u) &= -\frac{\sqrt{2}}{4}\sum_{i=1}^{N_f} n_i m_i + \frac{1}{2}\sqrt{2u}\left[1 + \sum_{i=1}^{\infty} a_{N_f,i}(\Lambda_{N_f}^{4-N_f}, m_1, \cdots, m_{N_f})u^{-i}\right], \\ a_D^{N_f}(u) &= -\frac{\sqrt{2}}{4}\sum_{i=1}^{N_f} n_i' m_i + i\frac{4-N_f}{2\pi}\tilde{a}_{N_f}(u)\ln\left(\frac{u}{\Lambda_{N_f}^2}\right) \\ &\quad + \sqrt{u}\sum_{i=0}^{\infty} a_{D_i}(\Lambda_{N_f}^{4-N_f}, m_1, \cdots, m_{N_f})u^{-i}, \end{aligned} \tag{5.1}$$

where $a_{N_f,i}(\Lambda_{N_f}^{4-N_f}, m_1, \cdots, m_{N_f})$ and $a_{D_i}(\Lambda_{N_f}^{4-N_f}, m_1, \cdots, m_{N_f})$ are homogeneous polynomials of order $2i$ and are invariant under the obvious $\boldsymbol{Z}_{N_f}$ symmetry. Then, from (5.1) we can easily find that the monodromy matrices $M_{N_f,\infty}$'s around $u = \infty$ act to the column vectors $v_{N_f} = {}^t(a_D^{N_f}, a_{N_f}, \epsilon_{N_f})$ as $v_{N_f} \longrightarrow M_{N_f,\infty} \cdot \nu_{N_f}$ and are given by

$$M_{N_f,\infty} = \begin{pmatrix} -1 & 4-N_f & \mu_{N_f} \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2}$$

where $\epsilon_{N_f} = n_1\nu_1 + \cdots + n_{N_f}\nu_{N_f}$, $\mu_{N_f} = 2n'_{N_f}/n_{N_f} - (4 - N_f)$. The non-trivial relations among the winding numbers are

$$\begin{aligned} &N_f = 1 && ------ \\ &N_f = 2 && n_1 n_2' = n_2 n_1', \\ &N_f = 3 && n_1 n_2 n_3' = n_1 n_2' n_3 = n_1' n_2 n_3. \end{aligned} \tag{5.3}$$

As for the prepotentials $\mathcal{F}_{N_f}$, we have established that they are given by the following simple formulae,

$$\begin{aligned} \mathcal{F}_{N_f} = i\frac{\tilde{a}_{N_f}^2}{\pi}\Bigg[&\frac{4-N_f}{4}\ln\left(\frac{\tilde{a}_{N_f}}{\Lambda_{N_f}}\right)^2 + \mathcal{F}_0^{N_f} - \frac{\sqrt{2}\pi}{4i\tilde{a}_{N_f}}\sum_{i=1}^{N_f} n_i' m_i + \frac{N_f}{2}\ln\tilde{a}_{N_f} \\ &+ \frac{1}{4\tilde{a}_{N_f}^2}\left(\frac{3}{2}\sum_{i=1}^{N_f} m_i^2 - \mathcal{F}_s^{N_f}\right) + \sum_{i=2}^{\infty}\widetilde{\mathcal{F}}_i^{N_f}(\Lambda_{N_f}^{4-N_f}, m_1, \cdots, m_{N_f})\tilde{a}_{N_f}^{-2i}\Bigg], \end{aligned} \tag{5.4}$$

where

$$\mathcal{F}_s^{N_f} = \sum_{i=1}^{N_f} \left(\tilde{a}_{N_f} - \frac{m_i}{\sqrt{2}}\right)^2 \ln\left(\tilde{a}_{N_f} - \frac{m_i}{\sqrt{2}}\right) + \sum_{i=1}^{N_f} \left(\tilde{a}_{N_f} + \frac{m_i}{\sqrt{2}}\right)^2 \ln\left(\tilde{a}_{N_f} + \frac{m_i}{\sqrt{2}}\right), \qquad (5.5)$$

and $\mathcal{F}_0^{N_f}$'s are some calculable constants independent of Λ_{N_f} and m_i. Here we have used $\tilde{a}_{N_f} = a_{N_f} - (n_1\nu_1 + \cdots + n_{N_f}\nu_{N_f})$. Note that $\widetilde{\mathcal{F}}_i^{N_f}$'s are homogeneous polynomials.

Finally, we comment on some open problems. Though we have restricted ourselves within the discussions in the weak coupling limit, it is important to quantitatively investigate in the strong coupling region. However, since there are many technical obstacles to accomplish it in the massive theories, it may be useful to do it by using some integrable systems[12,13]. If the investigations in the strong coupling region are done, we will be able to sufficiently understand the properties of these massive asymptotic free gauge theories. In addition to this, we should also check whether these formulae can be obtained by some field theoretical method.

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APPENDIX A EXPANSION COEFFICIENTS (I)

The first several coefficients of ρ_1 are

$$a_{2,0} = 1,$$

$$a_{2,1} = 0,$$

$$a_{2,2} = -\frac{1}{1024}(\Lambda_2^4 + 64\Lambda_2^2 m_1 m_2),$$

$$a_{2,3} = \frac{3\Lambda_2^4}{1024}(m_1^2 + m_2^2),$$

$$a_{2,4} = -\frac{15\Lambda_2^4}{4194304}(\Lambda_2^4 + 256\Lambda_2^2 m_1 m_2 + 4096 m_1^2 m_2^2),$$

$$
\begin{aligned}
a_{2,5} &= \frac{35\Lambda_2^6}{2097152}(m_1^2+m_2^2)(3\Lambda_2^2+128m_1m_2),\\
a_{2,6} &= -\frac{105\Lambda_2^6}{4294967296}\left[\Lambda_2^6+3072\Lambda_2^2(m_1^4+m_2^4)+576\Lambda_2^4m_1m_2+36864\Lambda_2^2m_1^2m_2^2\right.\\
&\quad\left.+262144m_1^3m_2^3\right].
\end{aligned}
\tag{A1}
$$

APPENDIX B EXPANSION COEFFICIENTS (II)

The first several coefficients of ρ_2 are

$$
\begin{aligned}
b_{2,1} &= \frac{1}{2}(m_1^2+m_2^2),\\
b_{2,2} &= \frac{1}{3072}\left[3\Lambda_2^4+256(m_1^4+m_2^4)\right],\\
b_{2,3} &= \frac{m_1^2+m_2^2}{30720}\left[15\Lambda_2^4+960\Lambda_2^2m_1m_2+1024(m_1^4-m_1^2m_2^2+m_2^4)\right],\\
b_{2,4} &= \frac{1}{58720256}\left[91\Lambda_2^8-71680\Lambda_2^4(m_1^4+m_2^4)+1048576(m_1^8+m_2^8)-3584\Lambda_2^6m_1m_2\right.\\
&\quad\left.+917504\Lambda_2^2(m_1^5m_2+m_1m_2^5-\Lambda_2^2m_1^2m_2^2)\right],\\
b_{2,5} &= \frac{m_1^2+m_2^2}{377487360}\left[1935\Lambda_2^8-215040\Lambda_2^4(m_1^4+m_2^4)+4194304(m_1^8+m_2^8)\right.\\
&\quad+687360\Lambda_2^6m_1m_2+3440640\Lambda_2^4m_1^2m_2^2+4194304(m_1^4m_2^4-m_1^2m_2^6-m_1^6m_2^2)\\
&\quad\left.+3932160\Lambda_2^2(m_1m_2^5+m_1^5m_2-m_1^3m_2^3)\right],\\
b_{2,6} &= \frac{1}{283467841536}\left[1793\Lambda_2^{12}-17554944\Lambda_2^8(m_1^4+m_2^4)-103809024\Lambda_2^4(m_1^8+m_2^8)\right.\\
&\quad-297792\Lambda_2^{10}m_1m_2+2214592512\Lambda_2^2(m_1^9m_2+m_1m_2^9)-146792448\Lambda_2^8m_1^2m_2^2\\
&\quad+1453326336\Lambda_2^4(m_1^6m_2^2+m_1^2m_2^6)-2860515328\Lambda_2^6m_1^3m_2^3\\
&\quad\left.-272498688\Lambda_2^6(m_1m_2^5+m_1^5m_2)+2147483648(m_2^{12}+m_1^{12})\right].
\end{aligned}
\tag{B1}
$$

APPENDIX C RESULTS OF THE MASSIVE $N_F = 3$ THEORY

In this appendix, we summarize the results for the massive $N_f = 3$ theory. We use the same notations as in the text, unless we mention especially.

In this theory, its quantum moduli space can be described by the following hyperelliptic curve

$$y^2 = F(x)^2 - G(x), \tag{C1}$$

and the meromorphic 1-form

$$\lambda_3 = \frac{\sqrt{2}x dx}{4\pi i y}\left[\frac{F(x)G'(x)}{2G(x)} - F'(x)\right], \tag{C2}$$

where

$$\begin{aligned} F(x) &= x^2 - u + \Lambda_3\left(\frac{m_1+m_2+m_3}{8} + \frac{x}{4}\right), \\ G(x) &= \Lambda_3(x+m_1)(x+m_2)(x+m_3). \end{aligned} \tag{C3}$$

The prime denotes the differentiation over x. Four branching points of (C1) in the weak coupling limit are given by

$$\begin{aligned} x_1 &= \frac{\Lambda_3}{8} + \sqrt{u} + \frac{\sqrt{\Lambda_3}}{2}u^{1/4} + \frac{\sqrt{\Lambda_3}}{4u^{1/4}}\left(m_1+m_2+m_3+\frac{\Lambda_3}{16}\right) \\ &\quad + \frac{\Lambda_3}{16\sqrt{u}}(m_1+m_2+m_3) + \cdots, \\ x_2 &= \frac{\Lambda_3}{8} + \sqrt{u} - \frac{\sqrt{\Lambda_3}}{2}u^{1/4} - \frac{\sqrt{\Lambda_3}}{4u^{1/4}}\left(m_1+m_2+m_3+\frac{\Lambda_3}{16}\right) \\ &\quad + \frac{\Lambda_3}{16\sqrt{u}}(m_1+m_2+m_3) + \cdots, \\ x_3 &= \frac{\Lambda_3}{8} - \sqrt{u} + i\frac{\sqrt{\Lambda_3}}{2}u^{1/4} - \frac{i\sqrt{\Lambda_3}}{4u^{1/4}}\left(m_1+m_2+m_3+\frac{\Lambda_3}{16}\right) \\ &\quad - \frac{\Lambda_3}{16\sqrt{u}}(m_1+m_2+m_3) + \cdots, \\ x_4 &= \frac{\Lambda_3}{8} - \sqrt{u} - i\frac{\sqrt{\Lambda_3}}{2}u^{1/4} + \frac{i\sqrt{\Lambda_3}}{4u^{1/4}}\left(m_1+m_2+m_3+\frac{\Lambda_3}{16}\right) \\ &\quad - \frac{\Lambda_3}{16\sqrt{u}}(m_1+m_2+m_3) + \cdots. \end{aligned} \tag{C4}$$

Then the two periods of λ_3 are defined by

$$a_3(u) = \oint_{\alpha'} \lambda_3,$$
$$a_D^3(u) = \oint_{\beta'} \lambda_3, \tag{C5}$$

where α and β are loops which may enclose the "extra" poles corresponding to $x = -m_1, -m_2, -m_3$. We can identify the canonical α-cycle with a loop which enclose the two branching points from x_4 to x_1 counter clockwise and the canonical β-cycle with a loop from x_1 to x_2 as well.

Using $\Pi_3 = \oint_\gamma \lambda_3$, where γ is a suitable 1-cycle on the curve, we can obtain the massive $N_f = 3$ Picard-Fuchs equation

$$\frac{d^3\Pi_3}{du^3} + \left(\frac{\Delta_3'}{\Delta_3} - \frac{4D_3}{B_3}\right)\frac{d^2\Pi_3}{du^2} - \frac{256}{\Delta_3}\left(C_3 - \frac{A_3D_3}{B_3}\right)\frac{d\Pi_3}{du} = 0, \tag{C6}$$

where

$$\begin{aligned}
A_3 &= \Lambda_3^4 K_1(K_3 - L_1 + S_1) + 2048u^2(3u^2 + N_2 - 2uK_2) \\
&\quad +4\Lambda_3^3[-2K_5 - 10S_1K_2 + 2S_1N_1 + 2L_3 - 2H_4 + u(K_3 + L_1 + 18S_1)] \\
&\quad +256\Lambda_3\Big[-u^3K_1 - 6S_1N_2 + 4S_2K_1 \\
&\quad +2u^2(K_3 + L_1 - 12S_1) - u(3L_3 + 3S_1N_1 - 14S_1K_2)\Big] \\
&\quad +32\Lambda_3^2\left[-5u^2K_2 + u(2K_4 + 18N_2 - 6S_1K_1) - 6L_4 + 2S_1K_3 + 2S_1L_1 - 6S_2\right], \\
B_3 &= 4K_2(\Lambda_3^2u + 16\Lambda_3S_1 + 128u^2) + 8N_2(\Lambda_3^2 - 96u) - S_1(\Lambda_3^3 + 192\Lambda_3u) \\
&\quad -256(u^3 - 4S_2) - 8\Lambda_3^2K_4, \\
C_3 &= \Lambda_3^3(K_3 + L_1 + 16S_1) + 8\Lambda_3^2(2K_4 - 6K_1S_1 + 16N_2 - 9uK_2) \\
&\quad +512(11u^3 - 2S_2 - 6u^2K_2 + 3uN_2) \\
&\quad +64\Lambda_3\left[-3L_3 - 3u^2K_1 + 12S_1K_2 + u(4K_3 + 4L_1 - 42S_1) - 3S_1N_1\right], \\
D_3 &= K_2(\Lambda_3^2 + 256u) - 48\Lambda_3S_1 - 192(u^2 + N_2), \\
\Delta_3 &= -1048576(u^5 + \Lambda_3S_3) - 32\Lambda_3^5S_1K_2 + 4096u^2(\Lambda_3^2 + 256K_2) \\
&\quad +\Lambda_3^6(K_4 - 2N_2) - 6144\Lambda_3^3S_1(2K_4 + N_2) - 12288\Lambda_3^2(9N_4 + 2S_2K_2)
\end{aligned}$$

$$
\begin{aligned}
&+128\Lambda_3^4(2K_6-13S_2-3L_4)+u^3\left[32768\Lambda_3^2K_2+131072(11\Lambda_3S_1-8N_2)\right]\\
&+u^2\left[2048S_1(-11\Lambda_3^3+512S_1)-128\Lambda_3^4K_2-1310720\Lambda_3S_1K_2\right.\\
&\left.-8192\Lambda_3^2(4K_4+23N_2)\right]+u\left[64\Lambda_3^5S_1+26624\Lambda_3^3S_1K_2\right.\\
&\left.+1179648\Lambda_3S_1N_2+256\Lambda_3^4(4N_2-K_4)+16384\Lambda_3^2(5S_2+9L_4)\right],
\end{aligned}
$$

$$
\begin{aligned}
S_i &= (m_1m_2m_3)^i,\\
K_i &= m_1^i+m_2^i+m_3^i,\\
L_i &= m_1^i(m_2^2+m_3^2)+m_2^i(m_3^2+m_1^2)+m_3^i(m_1^2+m_2^2),\\
N_i &= (m_1m_2)^i+(m_2m_3)^i+(m_3m_1)^i,\\
H_4 &= m_1^4(m_2+m_3)+m_2^4(m_3+m_1)+m_3^4(m_1+m_2).
\end{aligned}
\tag{C7}
$$

In the double scaling limit ($\Lambda_3 \longrightarrow 0, m_3 \longrightarrow \infty, \Lambda_3 m_3 = \Lambda_2^2$ fixed), they will be

$$
\begin{aligned}
A_3 &\longrightarrow A_{3\mathrm{dsl}} = 8m_3^2A_2,\\
B_3 &\longrightarrow B_{3\mathrm{dsl}} = -m_3^2B_2,\\
C_3 &\longrightarrow C_{3\mathrm{dsl}} = 16m_3^2C_2,\\
D_3 &\longrightarrow D_{3\mathrm{dsl}} = -2m_3^2D_2,\\
\Delta_3 &\longrightarrow \Delta_{3\mathrm{dsl}} = 256m_3^2\Delta_2.
\end{aligned}
\tag{C8}
$$

Using these coefficients, we can obtain (2.6) as a resultant of the double scaling limit.

The fundamental solutions to (C6) near $u=\infty$ ($z=1/u$) are given by

$$
\begin{aligned}
\tilde\rho_0(z) &= -\frac{\sqrt2}{4}\sum_{i=1}^{3} n_i m_i,\\
\tilde\rho_1(z) &= z^{-1/2}\sum_{i=0}^{\infty} a_{3,i}z^i,\\
\tilde\rho_2(z) &= \tilde\rho_1(z)\ln z + z^{-1/2}\sum_{i=1}^{\infty} b_{3,i}z^i,
\end{aligned}
\tag{C9}
$$

where

$$
\begin{aligned}
a_{3,0} &= 1,\\
a_{3,1} &= -\frac{\Lambda_3^2}{1024},\\
a_{3,2} &= -\frac{3\Lambda_3^4}{4194304} - \frac{\Lambda_3^2}{1024}(m_1^2+m_2^2+m_3^2) - \frac{\Lambda_3}{16}m_1m_2m_3,\\
a_{3,3} &= -\frac{5\Lambda_3^6}{4294967296} - \frac{3\Lambda_3^4}{2097152}(m_1^2+m_2^2+m_3^2) + \frac{3\Lambda_3^3}{16384}m_1m_2m_3\\
&\quad +\frac{3\Lambda_3^2}{1024}(m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2),\\
a_{3,4} &= -\frac{175\Lambda_3^8}{70368744177664} - \frac{15\Lambda_3^6}{4294967296}(m_1^2+m_2^2+m_3^2) + \frac{15\Lambda_3^5}{67108864}m_1m_2m_3\\
&\quad -\frac{15\Lambda_3^4}{1048576}\left[\frac{1}{4}(m_1^4+m_2^4+m_3^4) - (m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2)\right]\\
&\quad -\frac{15\Lambda_3^3}{16384}m_1m_2m_3(m_1^2+m_2^2+m_3^2) - \frac{15\Lambda_3^2}{1024}m_1^2m_2^2m_3^2
\end{aligned}
\tag{C10}
$$

and

$$
\begin{aligned}
b_{3,1} &= m_1^2+m_2^2+m_3^2+\frac{\Lambda_3^2}{512},\\
b_{3,2} &= \frac{\Lambda_3^4}{4194304} + \frac{\Lambda_3^2}{1024}(m_1^2+m_2^2+m_3^2) - \frac{\Lambda_3}{8}m_1m_2m_3 + \frac{1}{6}(m_1^4+m_2^4+m_3^4),\\
b_{3,3} &= -\frac{\Lambda_3^6}{12884901888} - \frac{\Lambda_3^4}{4194304}(m_1^2+m_2^2+m_3^2) - \frac{\Lambda_3^3}{4096}m_1m_2m_3\\
&\quad +\frac{\Lambda_3^2}{128}\left[\frac{3}{16}(m_1^4+m_2^4+m_3^4) + m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2\right]\\
&\quad +\frac{\Lambda_3}{16}m_1m_2m_3(m_1^2+m_2^2+m_3^2) + \frac{1}{15}(m_1^6+m_2^6+m_3^6),\\
b_{3,4} &= -\frac{265\Lambda_3^8}{422212465065984} - \frac{3\Lambda_3^6}{2147483648}(m_1^2+m_2^2+m_3^2) + \frac{\Lambda_3^5}{67108864}m_1m_2m_3\\
&\quad +\frac{\Lambda_3^4}{524288}\left[\frac{13}{16}(m_1^4+m_2^4+m_3^4) + 7(m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2)\right]\\
&\quad -\frac{\Lambda_3^2}{1024}\left[\frac{5}{2}(m_2^2+m_3^2)(m_1^2m_2^2+m_2^2m_3^2+m_1^4) + \frac{5}{6}(m_1^6+m_2^6+m_3^6) + 61m_1^2m_2^2m_3^2\right]\\
&\quad -\frac{31\Lambda_3^3}{16384}m_1m_2m_3(m_1^2+m_2^2+m_3^2) + \frac{\Lambda_3}{32}m_1m_2m_3(m_1^4+m_2^4+m_3^4)\\
&\quad +\frac{1}{28}(m_1^8+m_2^8+m_3^8).
\end{aligned}
\tag{C11}
$$

From the lower oredr expansion of the periods, we find that

$$
a_3(u) = \frac{\tilde{\rho}_1(z)}{\sqrt{2}} + \sum_{i=1}^{3} n_i\nu_i,
$$

$$a_D^3(u) = A'\tilde{\rho}_2(z) + B'\tilde{\rho}_1(z) + \sum_{i=1}^{3} n'_i \nu_i, \tag{C12}$$

where

$$\begin{aligned} A' &= -\frac{i\sqrt{2}}{4\pi}, \\ B' &= \frac{i\sqrt{2}}{4\pi}(8\ln 2 - 1 - \pi i - 2\ln\Lambda_3), \\ \nu_i &= -\frac{\sqrt{2}}{4} m_i. \end{aligned} \tag{C13}$$

Then the monodromy of the periods near $u = \infty$ will be

$$\begin{aligned} a_3 &\longrightarrow -a_3 + 2\sum_{i=1}^{3} n_i \nu_i, \\ a_D^3 &\longrightarrow -a_D^3 + a_3 + \sum_{i=1}^{3}(2n'_i - n_i)\nu_i. \end{aligned} \tag{C14}$$

Prepotential will be

$$\begin{aligned} \mathcal{F}_3 = i\frac{\tilde{a}_3^2}{\pi}\Bigg[&\frac{1}{4}\ln\left(\frac{\tilde{a}_3}{\Lambda_3}\right)^2 + \frac{1}{4}(9\ln 2 - 2 - \pi i) - \frac{\sqrt{2}\pi}{4i\tilde{a}_3}\sum_{i=1}^{3} n'_i m_i \\ &-\frac{\ln\tilde{a}_3}{4}\sum_{i=1}^{3} m_i^2 + \sum_{i=2}^{\infty}\mathcal{F}_i^3 \tilde{a}_3^{-2i}\Bigg], \end{aligned} \tag{C15}$$

where first few instanton expansion coefficients are given by

$$\begin{aligned} \mathcal{F}_2^3 &= -\frac{\Lambda_3^4}{67108864} - \frac{\Lambda_3^2}{8192}(m_1^2 + m_2^2 + m_3^2) - \frac{\Lambda_3}{64}m_1 m_2 m_3 + \frac{1}{96}(m_1^4 + m_2^4 + m_3^4), \\ \mathcal{F}_3^3 &= \frac{3\Lambda_3^4}{67108864}(m_1^2 + m_2^2 + m_3^2) + \frac{\Lambda_3^3}{65536}m_1 m_2 m_3 \\ &\quad + \frac{3\Lambda_3^2}{16384}(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) + \frac{1}{960}(m_1^6 + m_2^6 + m_3^6), \\ \mathcal{F}_4^3 &= -\frac{5\Lambda_3^8}{9007199254740992} - \frac{5\Lambda_3^6}{206158430208}(m_1^2 + m_2^2 + m_3^2) - \frac{7\Lambda_3^5}{536870912}m_1 m_2 m_3 \\ &\quad - \frac{5\Lambda_3^4}{67108864}\left[\frac{1}{4}(m_1^4 + m_2^4 + m_3^4) + 5(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2)\right] \\ &\quad - \frac{5\Lambda_3^3}{393216}m_1 m_2 m_3(m_1^2 + m_2^2 + m_3^2) - \frac{5}{32768}\Lambda_3^2 m_1^2 m_2^2 m_3^2 \\ &\quad + \frac{1}{5376}(m_1^8 + m_2^8 + m_3^8). \end{aligned} \tag{C16}$$

We can easily find that (C16) coincide with those of Ref.9 when the three hypermultiplets are mass-less and that these coefficients does not vanish in general while $\mathcal{F}^3_{2n+1}$ $(n > 0)$ vanish in the mass-less limit. Note that we can rewrite these coefficients by using $\widetilde{\mathcal{F}}^3_i$, although we do not rewrite them here.

APPENDIX D $N_F = 1$ PICARD-FUCHS EQUATION

In this appendix, we briefly summarize the $N_f = 1$ Picard-Fuchs equation. For more systematic explanations, see Ref.10. The curve and the meromorphic 1-form of the $N_f = 1$ theory is given by

$$y^2 = (x^2 - u)^2 - \Lambda_1^3(x + m_1), \tag{D1}$$

and

$$\lambda_1 = \frac{\sqrt{2}x dx}{4\pi i y}\left[\frac{x^2 - u}{2(x + m_1)} - 2x\right], \tag{D2}$$

respectively. Then the Picard-Fuchs equation is given by

$$\begin{aligned}
&\frac{d^3\Pi_1}{du^3} + \frac{3\Delta_1 + \Delta_1'(4m_1^2 - 3u)}{\Delta_1(4m_1^2 - 3u)}\frac{d^2\Pi_1}{du^2} \\
&-\frac{8[4(2m_1^2 - 3u)(4m_1^2 - 3u) + 3(3\Lambda_1^3 m_1 - 4u^2)]}{\Delta_1(4m_1^2 - 3u)}\frac{d\Pi_1}{du} = 0,
\end{aligned} \tag{D3}$$

where

$$\Delta_1 = 27\Lambda_1^6 + 256\Lambda_1^3 m_1^3 - 288\Lambda_1^3 m_1 u - 256 m_1^2 u^2 + 256 u^3, \tag{D4}$$

and $\Delta_1' = d\Delta_1/du$. Δ_1 is the discriminat of the curve. Introducing following notations

$$A_1 = -4u^2 + 3m_1\Lambda_1^3,\ B_1 = -12u + 16m_1^2,\ C_1 = -3u + 2m_1^2,\ D_1 = -3, \tag{D5}$$

we can write (D3) more simply as

$$\frac{d^3\Pi_1}{du^3} + \left(\frac{\Delta_1'}{\Delta_1} - \frac{4D_1}{B_1}\right)\frac{d^2\Pi_1}{du^2} - \frac{32}{\Delta_1}\left(C_1 - \frac{A_1 D_1}{B_1}\right)\frac{d\Pi_1}{du} = 0, \tag{D6}$$

where the prime denotes the differentiation over u.

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